An Introduction to Multi-objective Optimization

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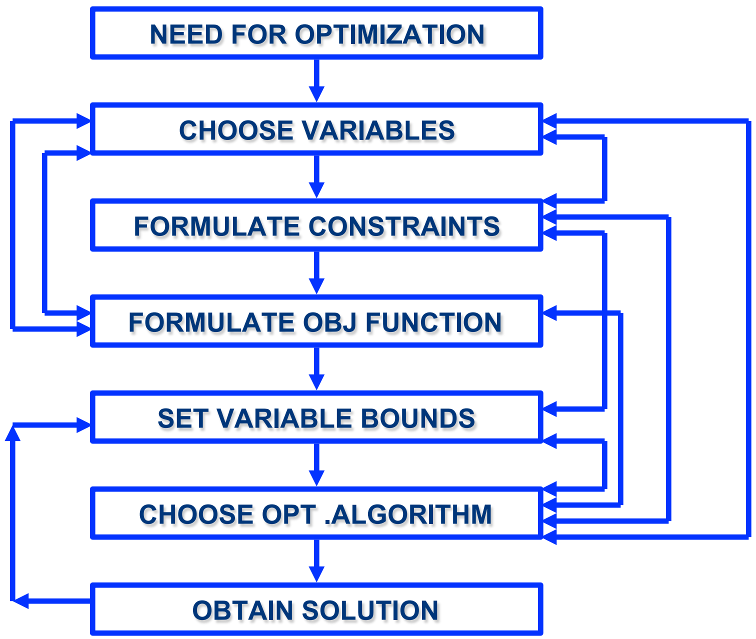
Optimization

- Act of obtaining the best result under given circumst ances

OPTIMAL DESIGN PROCEDURE



Single Vs Multi-objective

Single Objective Optimization:
When an optimization problem involves only one objective functio n, the task of finding the optimal solution is called single-objective optimization.

Example: Find out a **CAR** for me with Minimum cost.

Multi-objective Optimization: When an optimization problem in volves more than one objective function, the task of finding one or more optimal solutions is known as multi-objective optimization.

Example: Find out a **CAR** for me with minimum cost and maxim um comfort.

Examples on Multiple Objectives

- **Car design:**
  - **Need to reduce drag since this impacts on petrol consumption.**
  - **Needs to be able to accommodate people (!!!) and provide reas onable amount of other carrying space, e.g. for luggage.**
  - **Needs to have an aesthetic appeal (but people's tastes differ).**
  - **Performance related factors: speed, acceleration and weight of vehicle.**
  - **Strength…but also ability to absorb impacts.**
  - **And so on…**

So what is the general problem ?

- The Multiobjective optimization problem (M OP) can be defined as the problem of finding [Osyczka 1985] a vector of decision variables w hich satisfies constraints and optimizes a vect or function show elements represent the objec tive functions.
- Hence, the term “optimize” means finding suc h a solution which would give the values of all objective functions acceptable to the designer.

Mathematical formulation

SOOP AND MOOP

Opt (f(x))

subject to : $g_i(x) \leq 0$

where $i = 1,2,\ldots,q$

$h_i(x) = 0$

where $i = q+1,q+2,\ldots,m$

Opt (f(x))

$f(x) = (f_1(x), f_2(x),\ldots,f_k(x))$

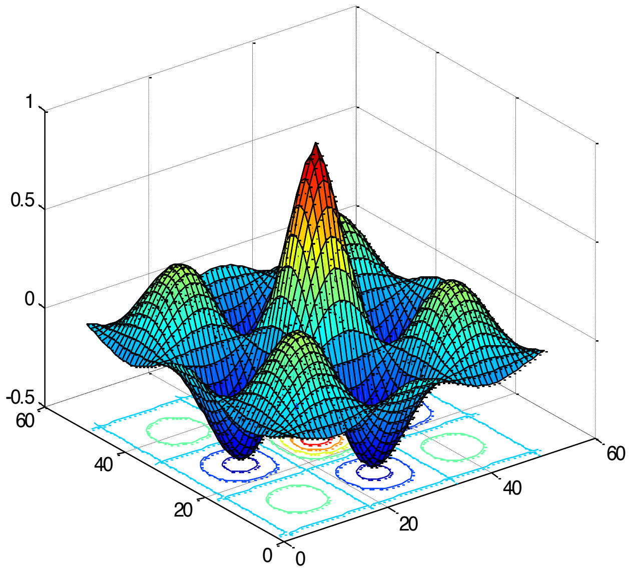
subject to : $g_i(x) \leq 0$

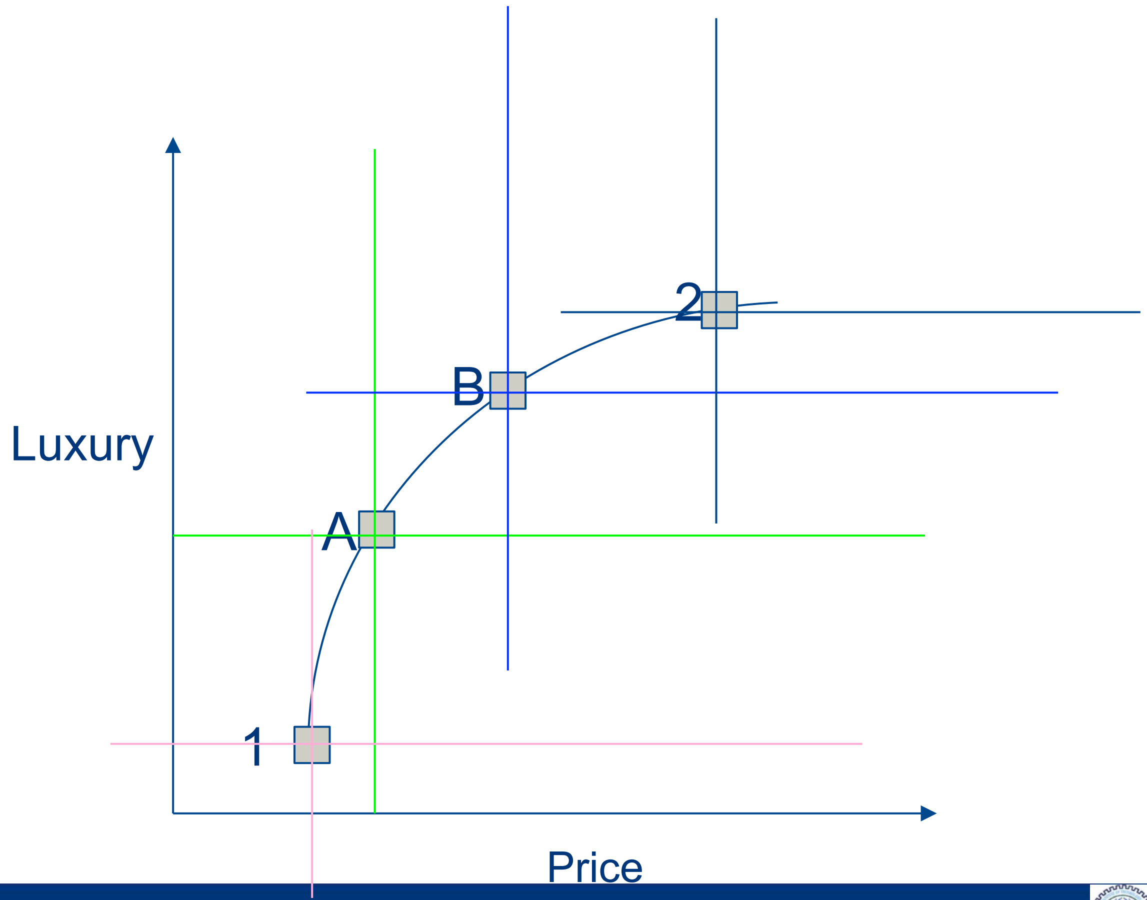
where $i = 1,2,\ldots,q$

$h_i(x) = 0$

where $i = q+1,q+2,\ldots,m$

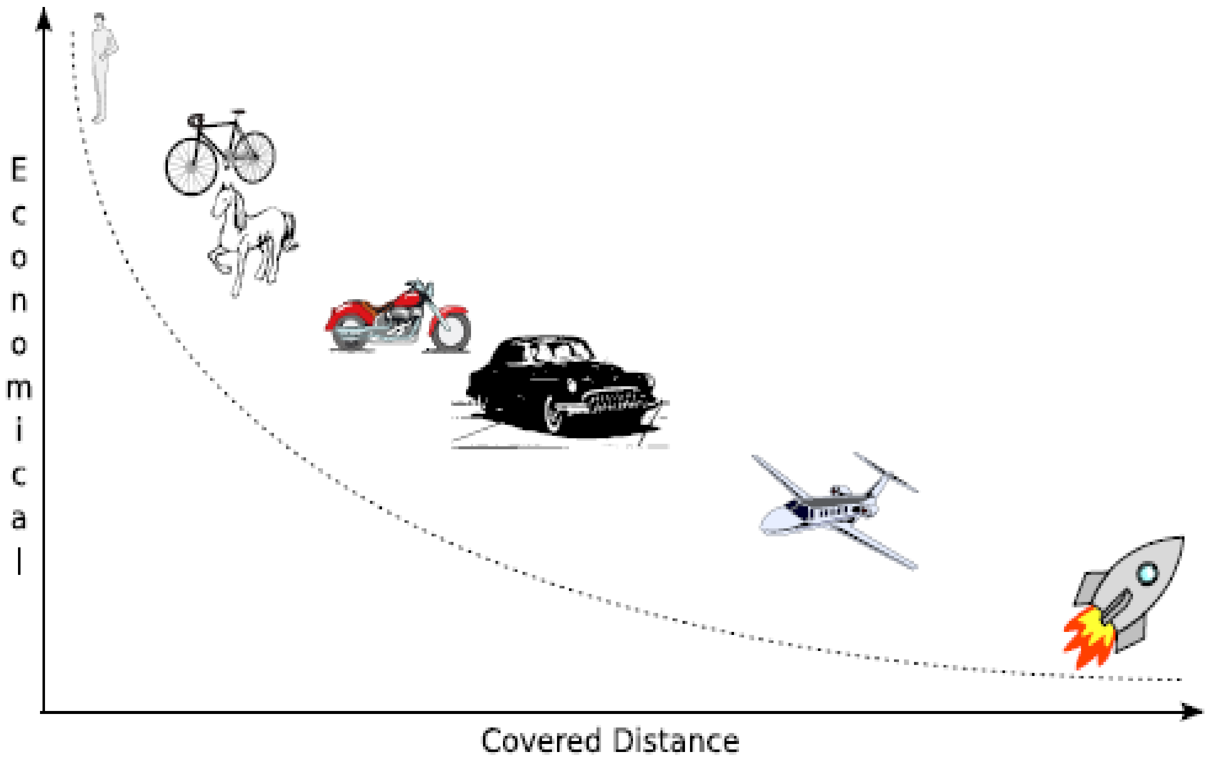
Single Vs Multi-objective: A Simple Visualization





Another example

Which means of transport we should choose?
It might depend on how far we need to go or how chea p we need it to be as shown in Figure



In single objective optimization superiority of a solution is determined by the comparison o f their objective function values

In multi-objective optimization goodness of t he solution is determined by the dominance

- **Multi-objective optimization (MOO) is the optimization of conflicting objectives.**
- **Suppose you need to fly on a long trip: Should you choose the cheapest ticket (more connections) or shortest flying time (more expensive)?**
- **Also, the relative importance will vary.**
  - **There may be an emergency you need to go fix quickly.**
  - **Or, maybe you are on a very tight budget.**

Pareto-Optimal Solutions Example

- **Suppose for our airplane-trip, we find the followin g tickets:**

Ticket	Travel Time (hrs)	Ticket Price ($)
A	10	1700
B	9	2000
C	8	1800
D	7.5	2300
E	6	2200

Comparison of Solutions

- **If we compare tickets A & B, we can't say that either is superior without knowing the relative importance of Travel Time vs. Price.**

- **However, comparing tickets B & C shows that C is better than B in both objectives, so we can say that C "*dominates*" B.**

- **So, as long as C is a feasible option, there is no reason we would choose B.**

Comparison of Solutions

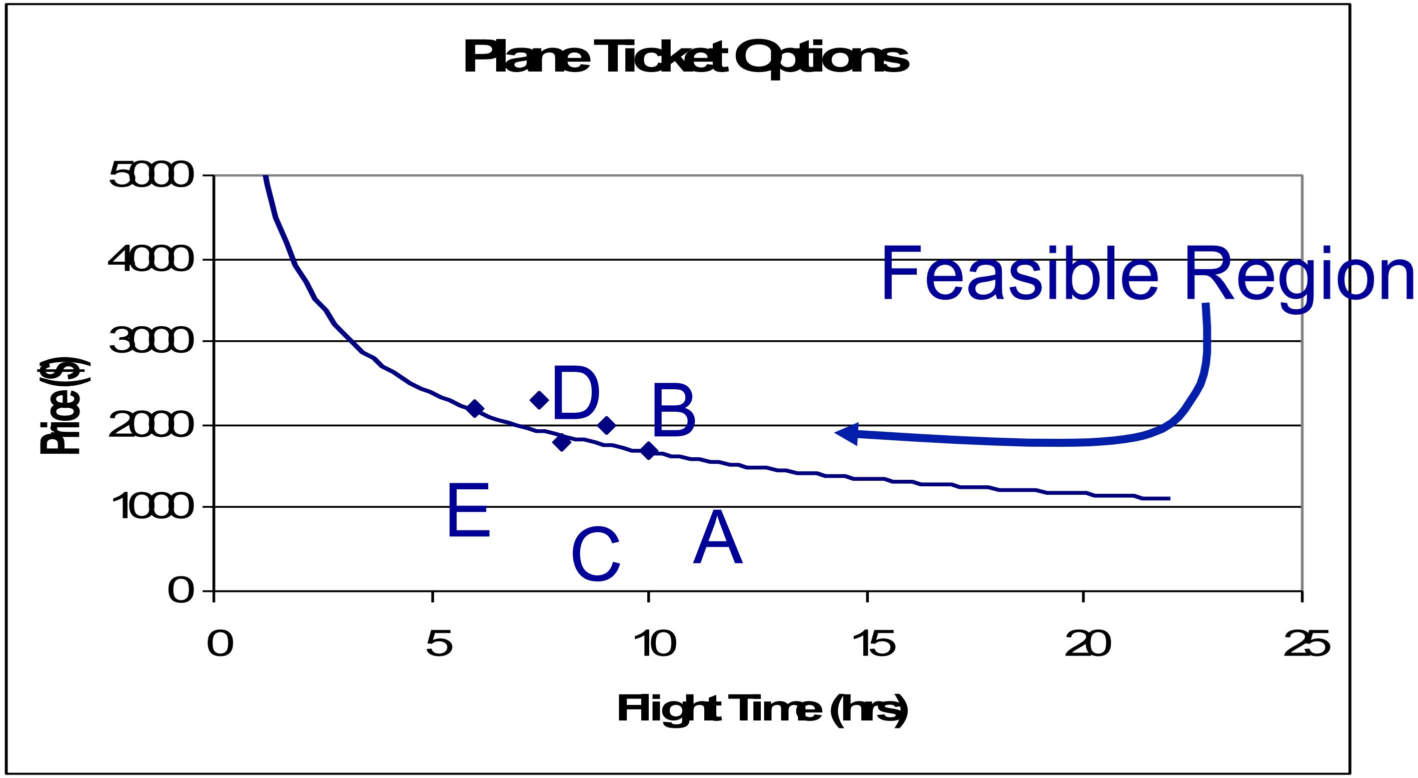
- **If we finish the comparisons, we also see that D is dominated by E.**

- **The rest of the options (A, C, & E) have a trad e-off associated with Time vs. Price, so none is clearly superior to the others.**

- **We call this the *"non-dominated"* set of solutio ns become none of the solutions are dominate d.**

Graph of Solutions

- **Usually, solutions of this type form a typical shape, shown in the chart below:**



Types of Solutions

- **Solutions that lie along the line are non-dominated solutions while those that lie inside the line are dominated because ther e is always another solution on the line that has at least one o bjective that is better.**

- **The line is called the Pareto front and solutions on it are calle d Pareto-optimal.**

- **All Pareto-optimal solutions are non-dominated.**

- **Thus, it is important in MOO to find the solutions as close as possible to the Pareto front & as far along it as possible.**

Pareto Optimum: Definition

- **A candidate is Pareto optimal iff:**
  - **It is at least as good as all other candidates for all objectives, and**
  - **It is better than all other candidates for at least one objective.**
- **We would say that this candidate *dominates* all oth er candidates.**

Mathematically

Given the vector of objective functions $\vec{f}(\vec{x}) = (f_1(\vec{x}), \ldots, f_k(\vec{x}))$

we say that candidate $\vec{x}_1$ dominates $\vec{x}_2$, if:

$$f_i(\vec{x}_1) \leq f_i(\vec{x}_2) \quad \forall i \in \{1, \ldots, k\}$$

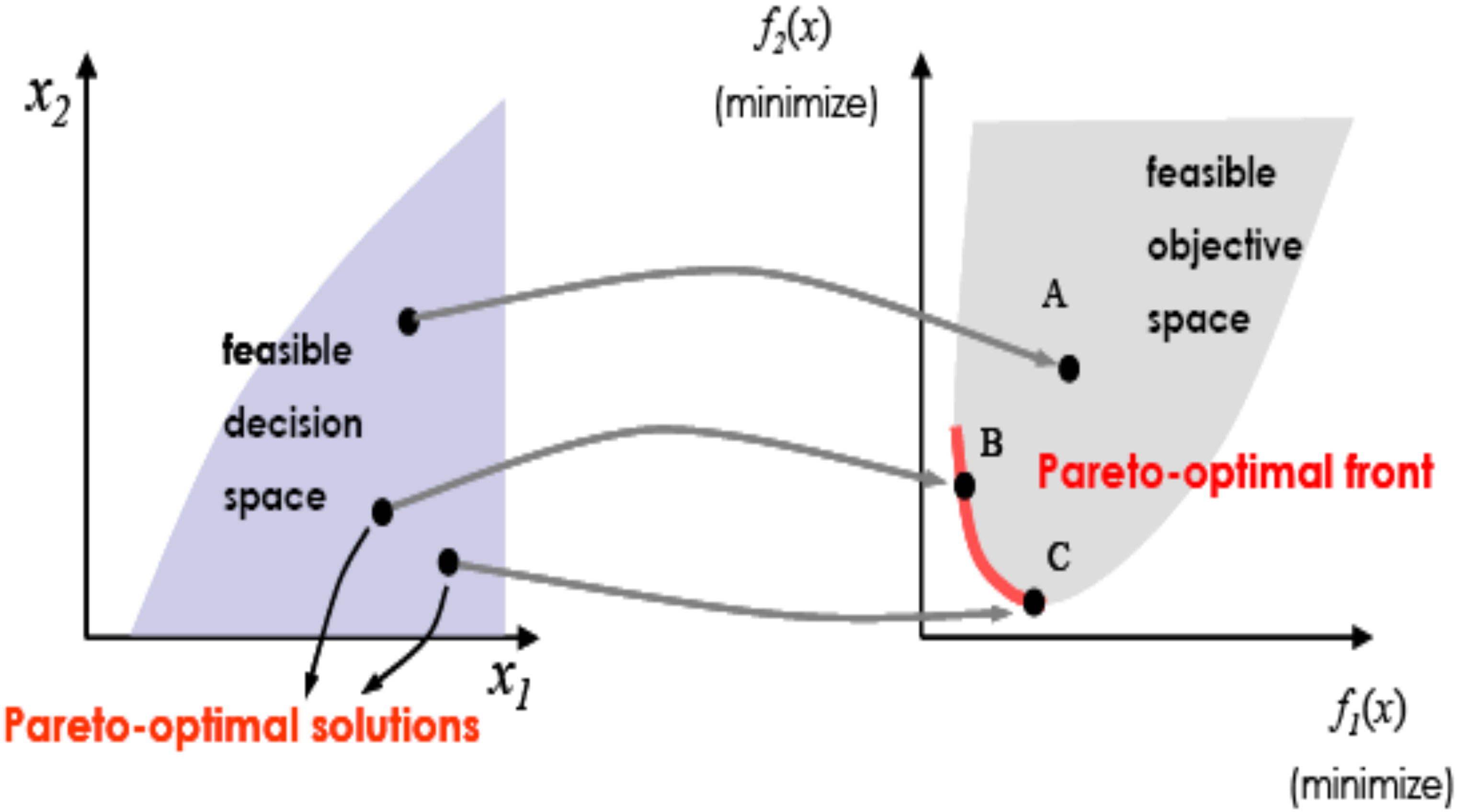
$$and$$

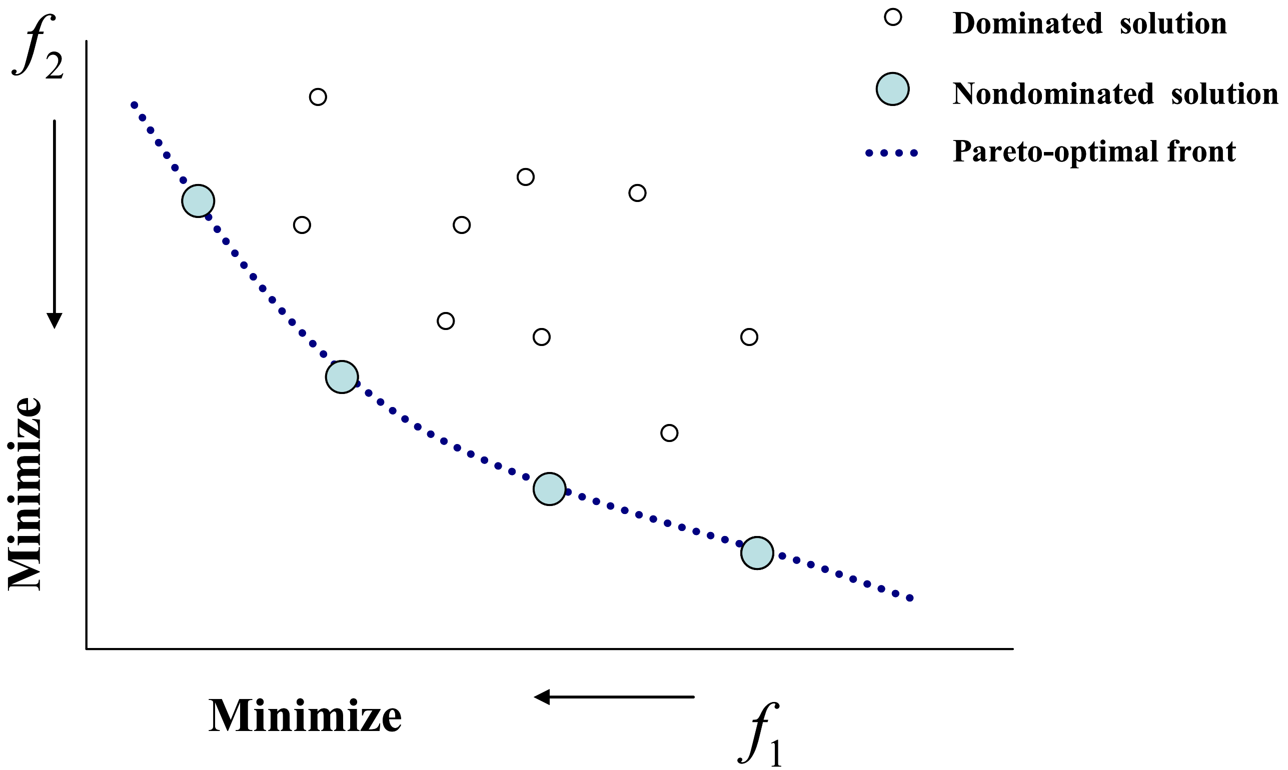
$$\exists i \in \{1, \ldots, k\} : f_i(\vec{x}_1) < f_i(\vec{x}_2)$$

(assuming we are trying to minimize the objective functions).

(Coello Coello 2002)

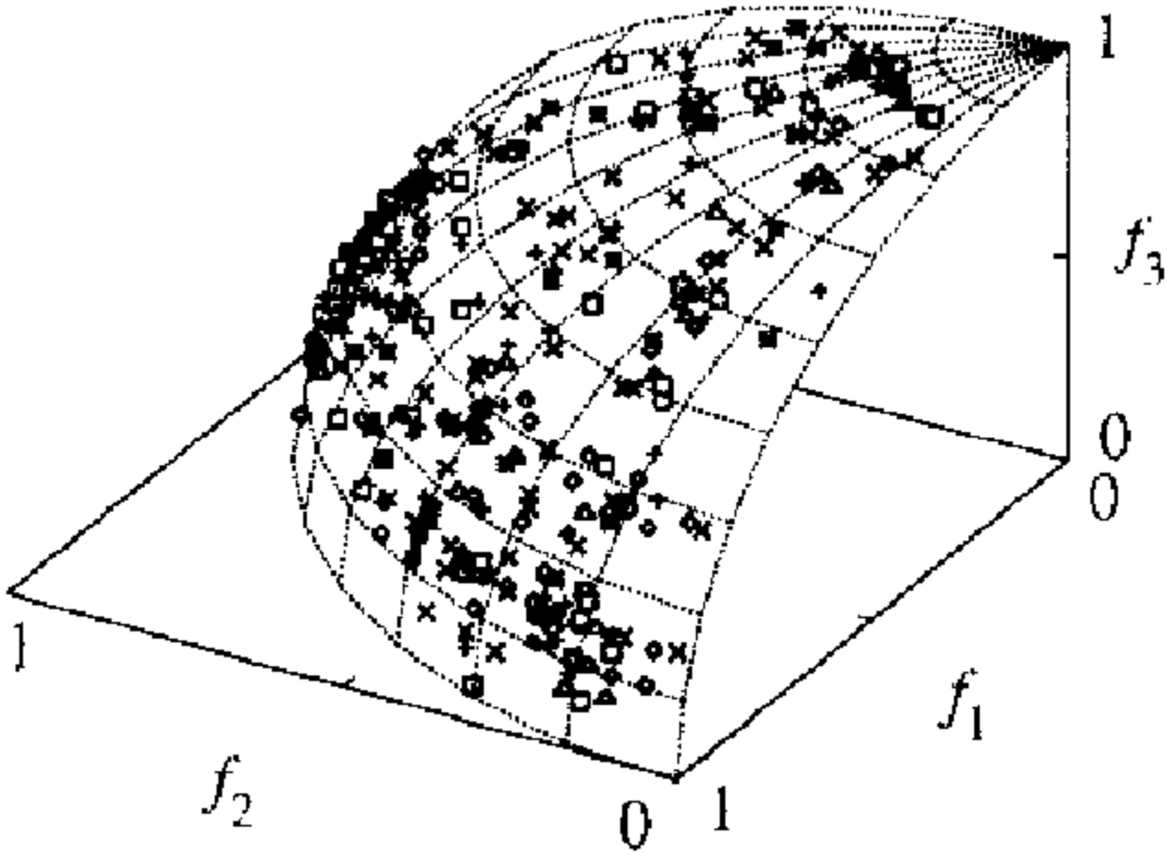
Graphical Depiction of Pareto Optimal Solution





f_2
Minimize
Minimize
f_1
Dominated solution
Nondominated solution
Pareto-optimal front

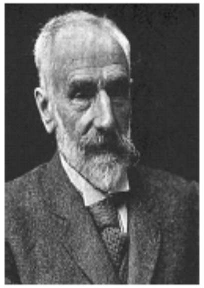
Pareto Front



(Tamaki *et al.* 1996)

What does optimum mean here?

- Having several objective functions implies that we are trying to find a good compromise rather than a single optimal solution.



- Francis Ysidro Edgeworth first proposed a meaning for "optimum" in 1881 which was generalized in 1896 by Vilfredo Pareto

- The concept of optimizing one performance on the cost of other is termed as Pareto optimality.

objective functions.

- The trade-off curve is also said to be Pareto optimal front and the points over it are termed as Pareto optimal points.

Pareto Principle

- "The Vital Few and Trivial Many Rule"

While the rule is not an absolute, one should use it as a guid e and reference point to ask whether or not you are truly fo cusing on:

20% - The Vital Few

or

80% - The Trivial Many

True progress results from a consistent focus on the 20% m ost critical objectives.

Pareto Principle

- **A small number of causes are responsible for a large percentage of the effect-**

  -usually a 20-percent to 80-percent ratio.

- **This basic principle translates well into quality prob lems - most quality problems result from a small nu mber of causes.**

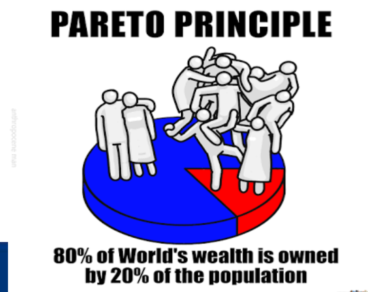
- **You can apply this ratio *to almost anything*, from the science of management to the physical world**

Addressing the most troublesome 20% of the problem will solve 80% of it.

Within your process, 20% of the individuals will cause 80% of your headaches.

Of all the solutions you identify, about 20% are likely to remain viable after adequate analysis.

80% of the work is usually done by 20% of the people.



80% of the quality can be gotten in 20% of the time -- perf ection takes 5 times longer

20% of the defects cause 80% of the problems.

Project Managers know that 20% of the work (the first 10 % and the last 10%) consume 80% of the time and resourc es.

Different Pareto Optimal Fronts

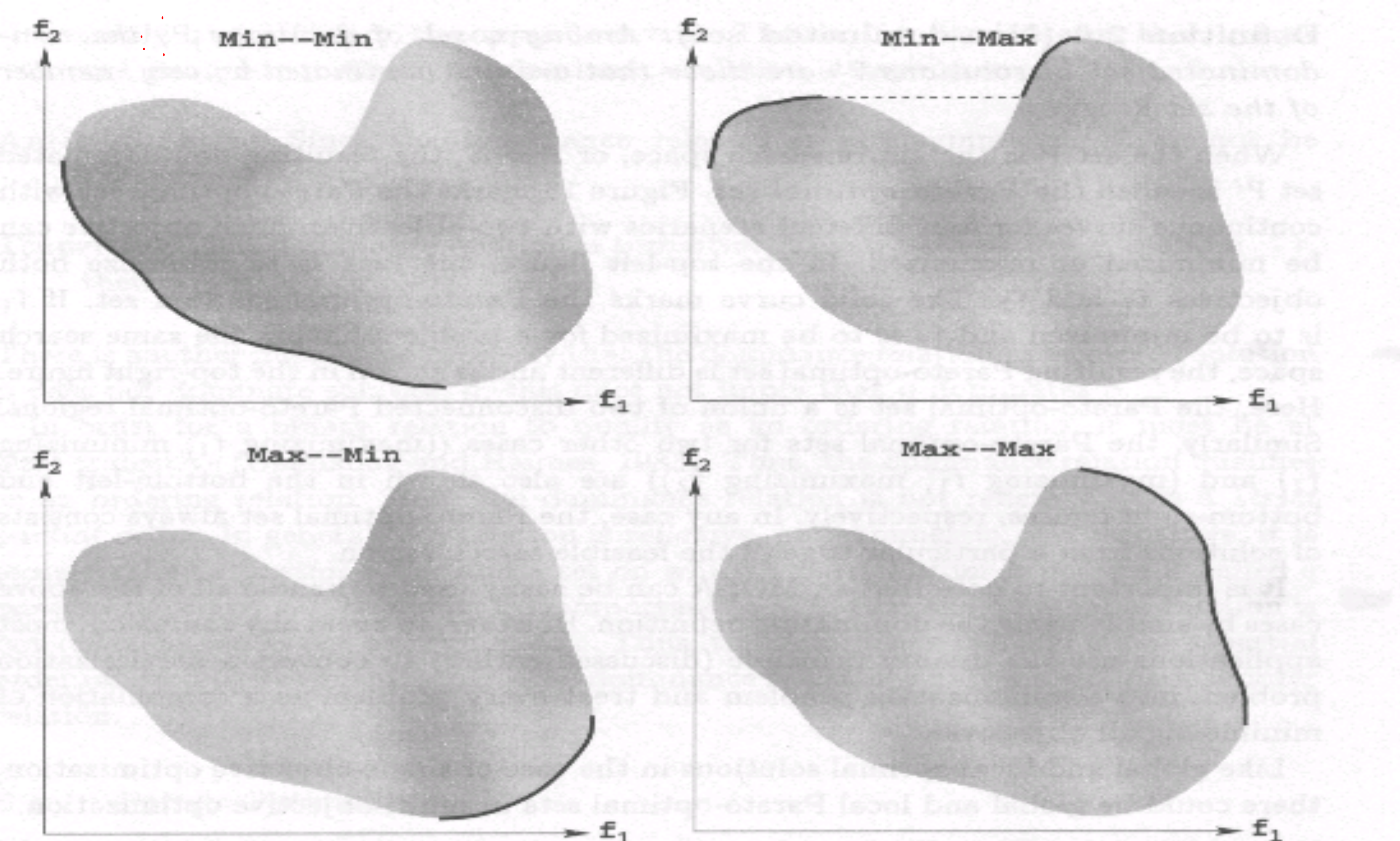
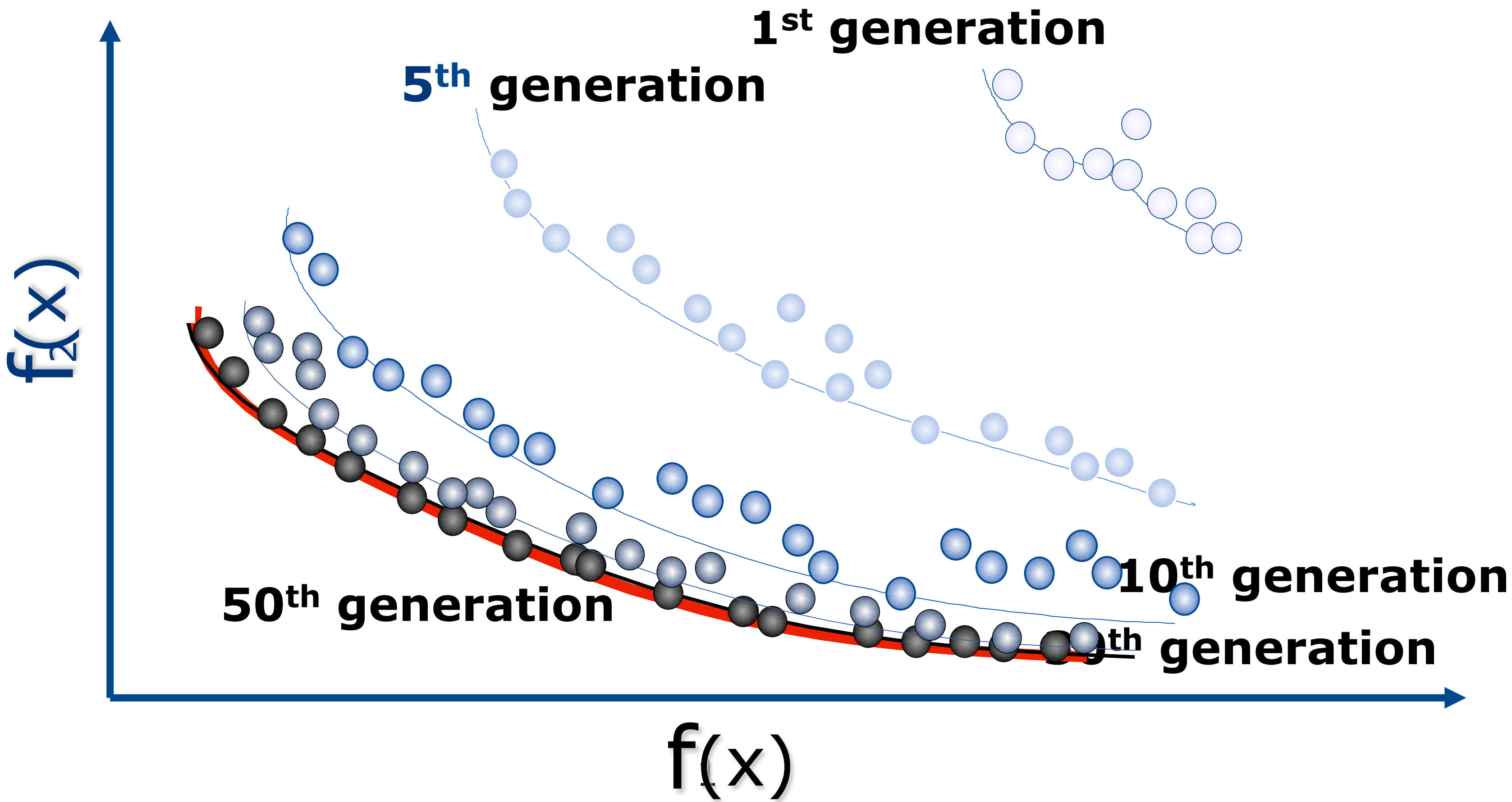


Figure from Deb p. 32

MOEA



1st generation
5th generation
10th generation
50th generation
generation
$f_2(x)$
f(x)

Why Evolutionary Approach to MOOP ?

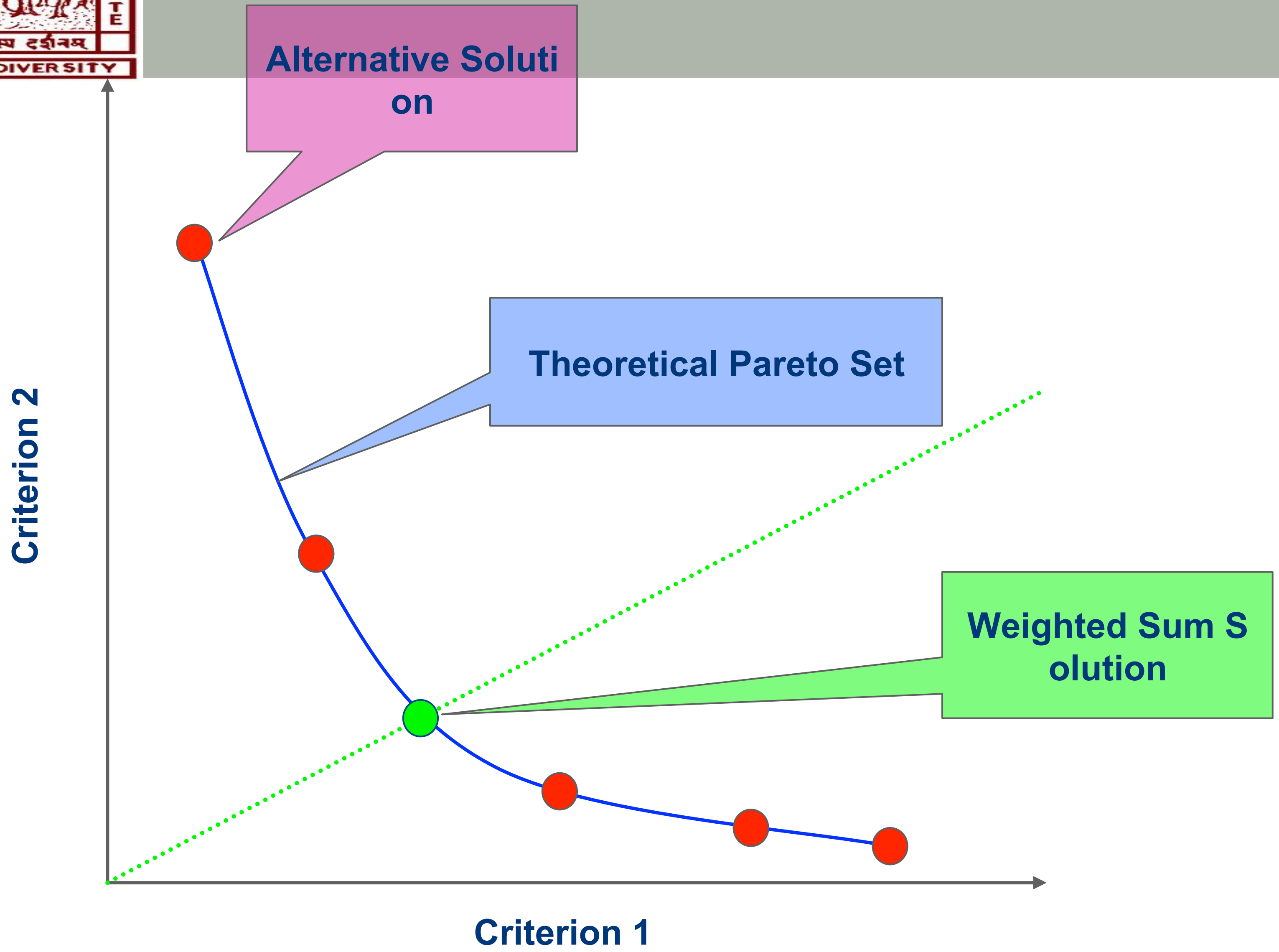
- **The major part of earlier mathematical research has concentrated on optimizatio n problems where the functions were linear, differentiable, convex, or otherwise mathematically well behaving.**
- **However, in practical problems, objective functions are often nonlinear, non-diffe rentiable, discontinuous, multi-modal etc. and no presumptions can be made abo ut their behaviour.**
- **Most traditional optimization methods cannot handle such complexity or do not perform in some cases in which the assumptions, upon which they are based do n ot hold.**
- **For such problems, stochastic optimization methods such as EAs have been imple mented effectively because they do not rely upon assumptions concerning the obj ective and constraint functions.**
- **EAs are stochastic search and optimization heuristic derived from the classic evo lution theory, working on a population of potential solutions to a problem. The b asic idea is that if only those individuals reproduce, which meet a certain selectio n criteria, the population will converge to solutions that best meet the selection cr iteria**

Classification of MOEA

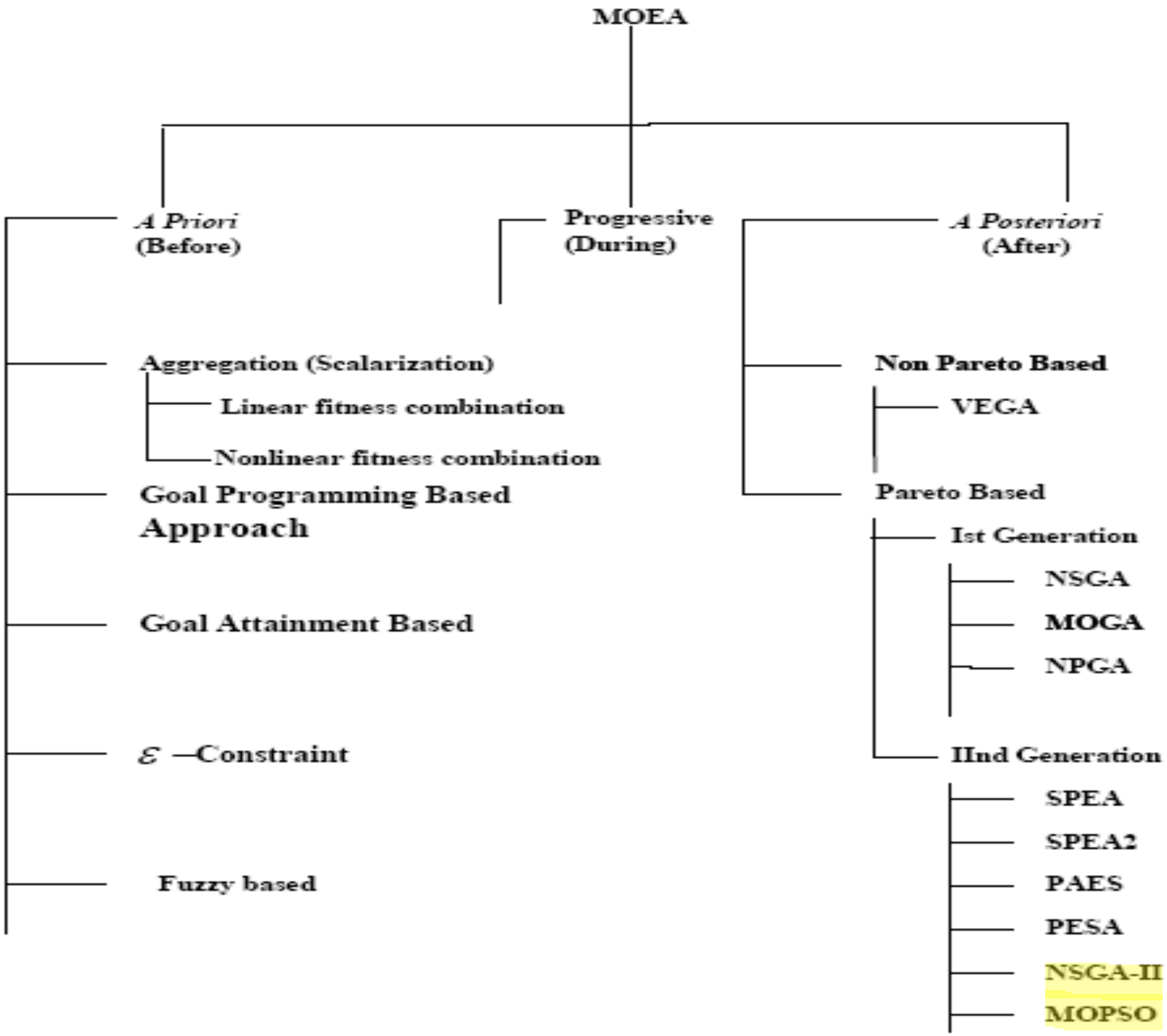
Hwang and Masud (1979) and later Miettinen (1999) fine-tuned the earlier classifications and suggested the following three main classes based on the preference articulation of the DM:

- ***A Priori* Preference Articulation:**
- **Progressive Preference Articulation:**
- ***A Posteriori* Preference Articulation:**

Your Answer


Alternative Soluti
on
Theoretical Pareto Set
Weighted Sum S
olution
Criterion 2
Criterion 1

Classification of MOEA contd…



Priori (Weighted Sum Approach)

Optimize

F(x)= $$\sum_{m=1}^{M} w_m . f_m(x) \qquad w_m \in [0,1], \sum_{m=1}^{M} w_m = 1$$

As we converted it into single objective we can now proceed using any EA with its associated operators.

Hopefully we will get an optimal solutions.

Problem: How to fix these weights? (Static/Dynamic)

Advantages and Disadvantages

- Efficient and easy to implement
- It does not have an explicit mechanism to maintain diversity.
- It doesn’t necessarily produce non-dominated vectors.
- No consideration of multi-objective nature of problem
- Each run will lead to only one final solution point (one point on Pareto front)
- Can not handle non-convexities of the Pareto front
- It is difficult to set the weight vectors

Posteriori Pareto Approach from EA Domain

VEGA (Vector Evaluated Genetic Algorithms)
(Contributed by David Schaffer in 1984).

VOES(Vector Optimized Evolution Strategy) contributed by Frank Kursawe in 1990.

MOGA (Multi-objective GA) introduced by Fonseca and Fleming in 1993.

NSGA (Non-dominated Sorting GA) introduced by Srinivas and Deb in 1994.

NPGA (Niched-Pareto Genetic Algorithm) introduced by Horn et al. in 1994.

PPES (Predator-Prey Evolution Strategy) introduced by Laumanns et al. in 1 998.

Nash GA introduced by Sefrioui and Periaux in 2000, motivated by a game th eoretic approach.

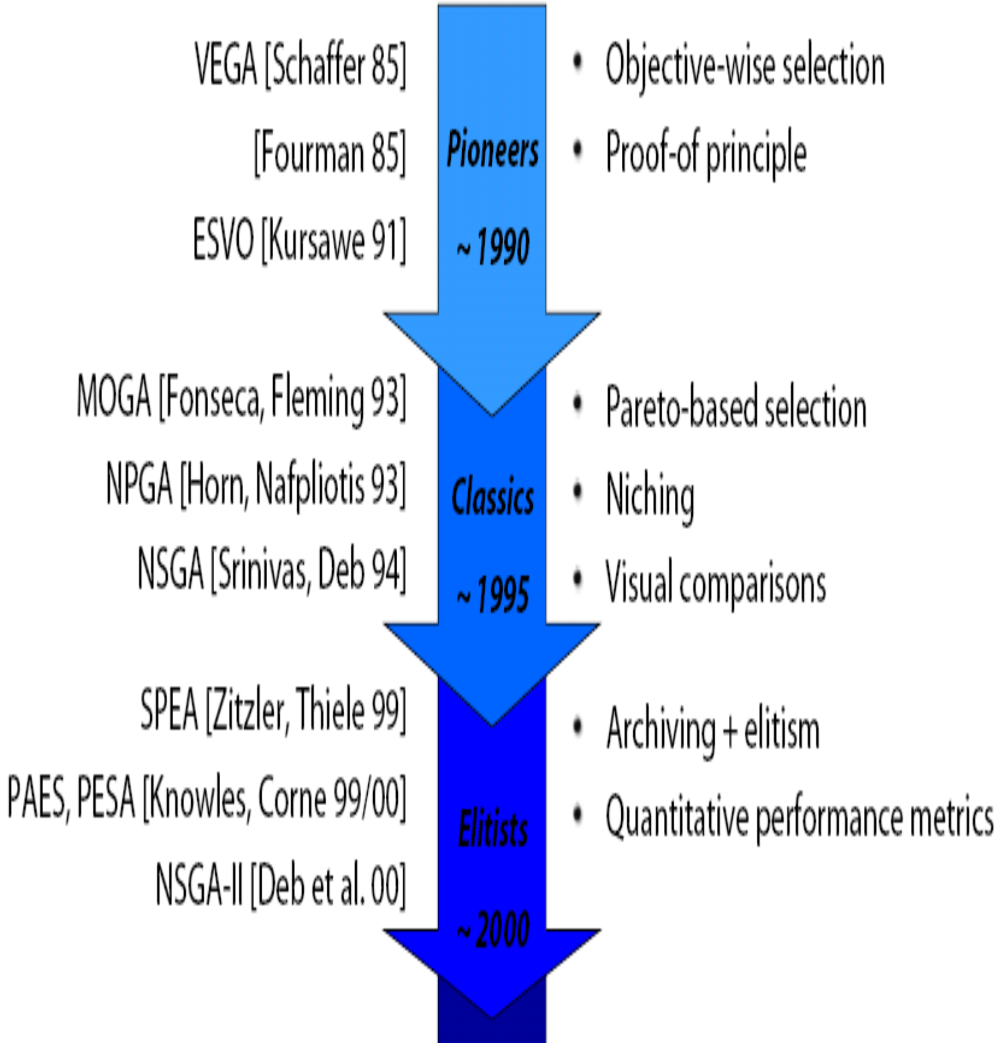
REMOEA (Rudolph's Elitist MOEA) introduced by Rudolph in 2001.

NSGA-II by Deb et a. in 2000.

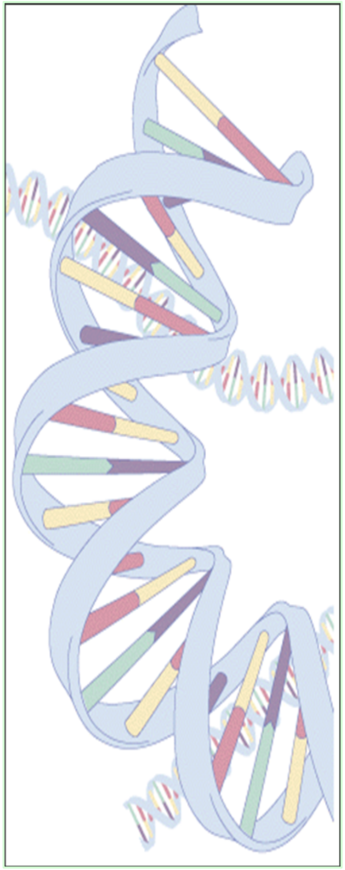
MOPSO by C. Coello and so on……………………..

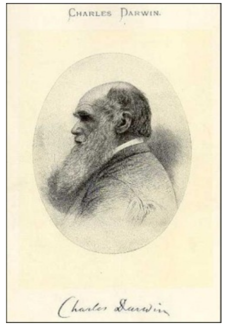
Classifying EMOO approaches
(Evolutionary Multi-Objective Optimization)

- **First Generation**
  - **Non-Pareto approaches**
  - **Pareto approaches**
- **Second Generation**
  - **PAES**
  - **SPEA**
  - **NSGA-II**
  - **MOMGA**
  - **micro-GA**



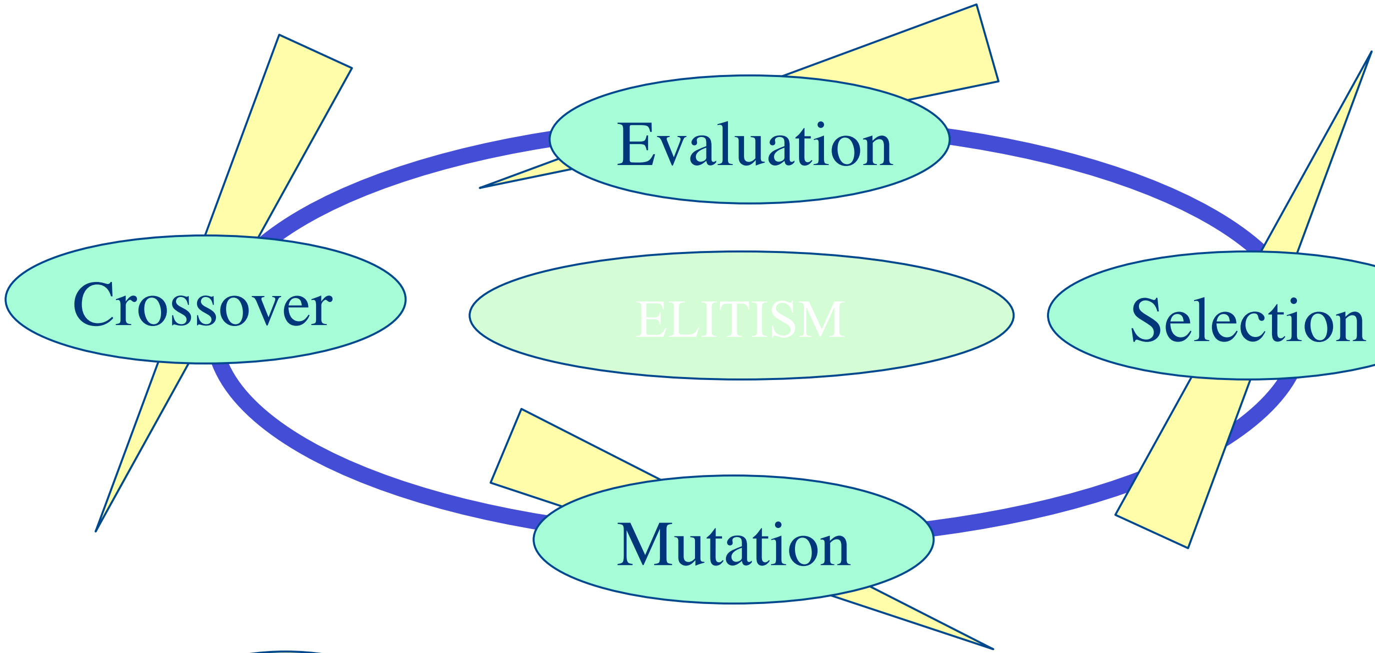
What are Genetic Algorithms ?

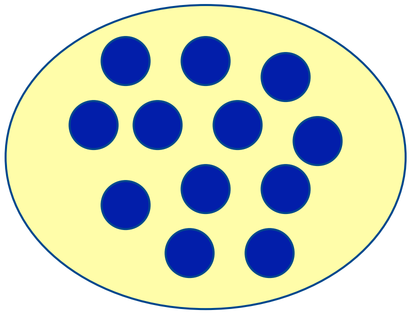




- Bio-Inspired artificial intelligence class of probabilistic optimization algorithms
- Developed by John Holland (1975)
- Influenced by Darwin's Origin of species (Survival of the fittest)
- Well-suited for nonlinear/hard problems with a large search space

GENETIC ALGORITHMS





Multi point searching methods

WHY USE GA ?

- They perform well in problems for which the fitness landscape is complex - ones
- Where the fitness function is noisy, changes over time, or has many local optima.
- Knows nothing about the problems they are deployed to solve.

Non Dominated Sorting based Genetic Al gorithm II (NSGA-II)

- **Developed at KanGAL (Prof K. Deb)**
- **Superior to most of the MOEA in the research arena today**
- **Uses *Elitism***
- **Famous for *Fast non-dominated search***
- **There were some major drawbacks in NSGA such as**
  - **High computational complexity of nondominated sorting**
  - **Lack of elitism**
  - **Lack of specification of sharing parameter**
  - **Deb et al. proposed an improved version of NSGA, called NS GA-II which dealt all the drawbacks of original NSGA**

Let P and Npop be the current population in NSGA-II and the population size, respectively (i.e. $Npop = |P|$). Then the outl

□ : P := **Initialize** (P)

□ : **do, until termination condition**

□ : P' :**Selection** (P)

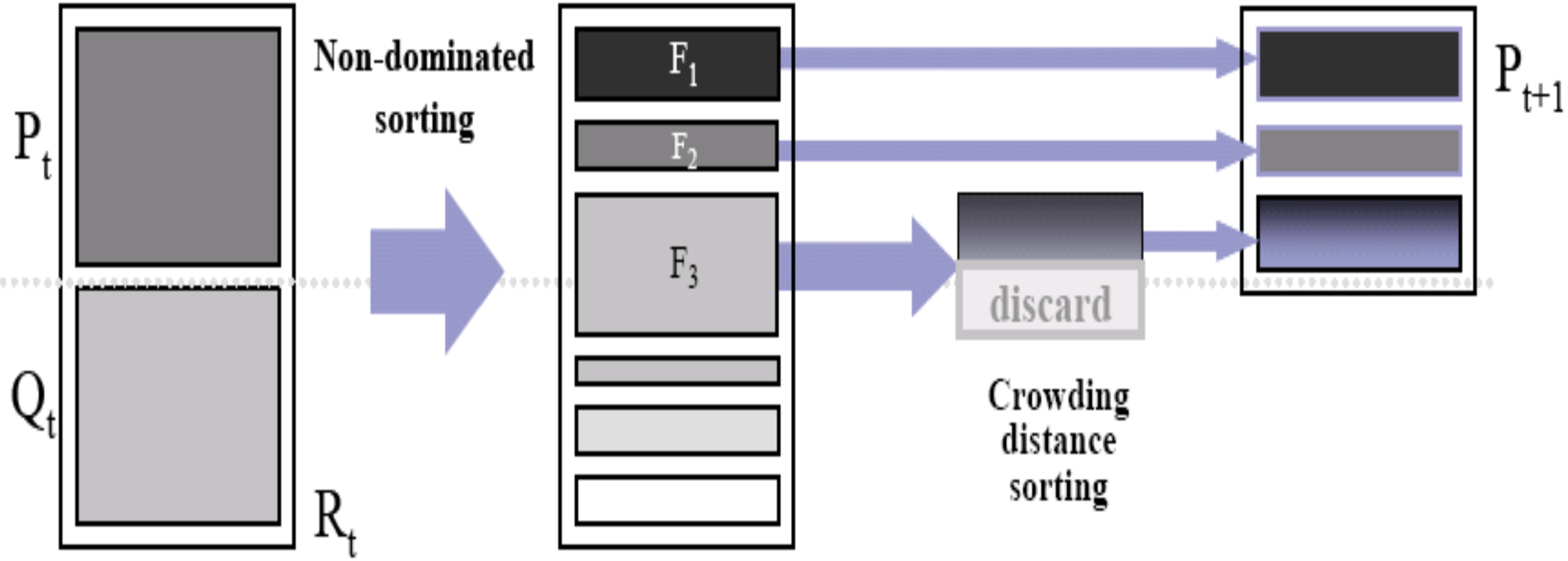
□ : P'' :**Genetic Operations** P'

□ : P := **Replace** ($P \cup P''$)

□ : **end while**

□ : **return (non-dominated solution** (P)

e to choose parent solution.



P_t
Q_t
R_t
Non-dominated
sorting
F_1
F_2
F_3
discard
Crowding
distance
sorting
P_{t+1}

Important Definitions

A general MOOP formulation in standard form is as follows:

$$\underset{x \in R^n}{Minimize}\,(Maximize)\ \ f(x) = \{f_1(x), f_2(x), \ldots, f_k(x)\} \qquad (1.0)$$

$$subject\ \ to: g_j(x) = 0, \quad j = 1, \ldots, m_e; \qquad (1.1)$$

$$g_j(x) \le 0, \quad j = m_e + 1, \ldots, m, \qquad (1.2)$$

where,

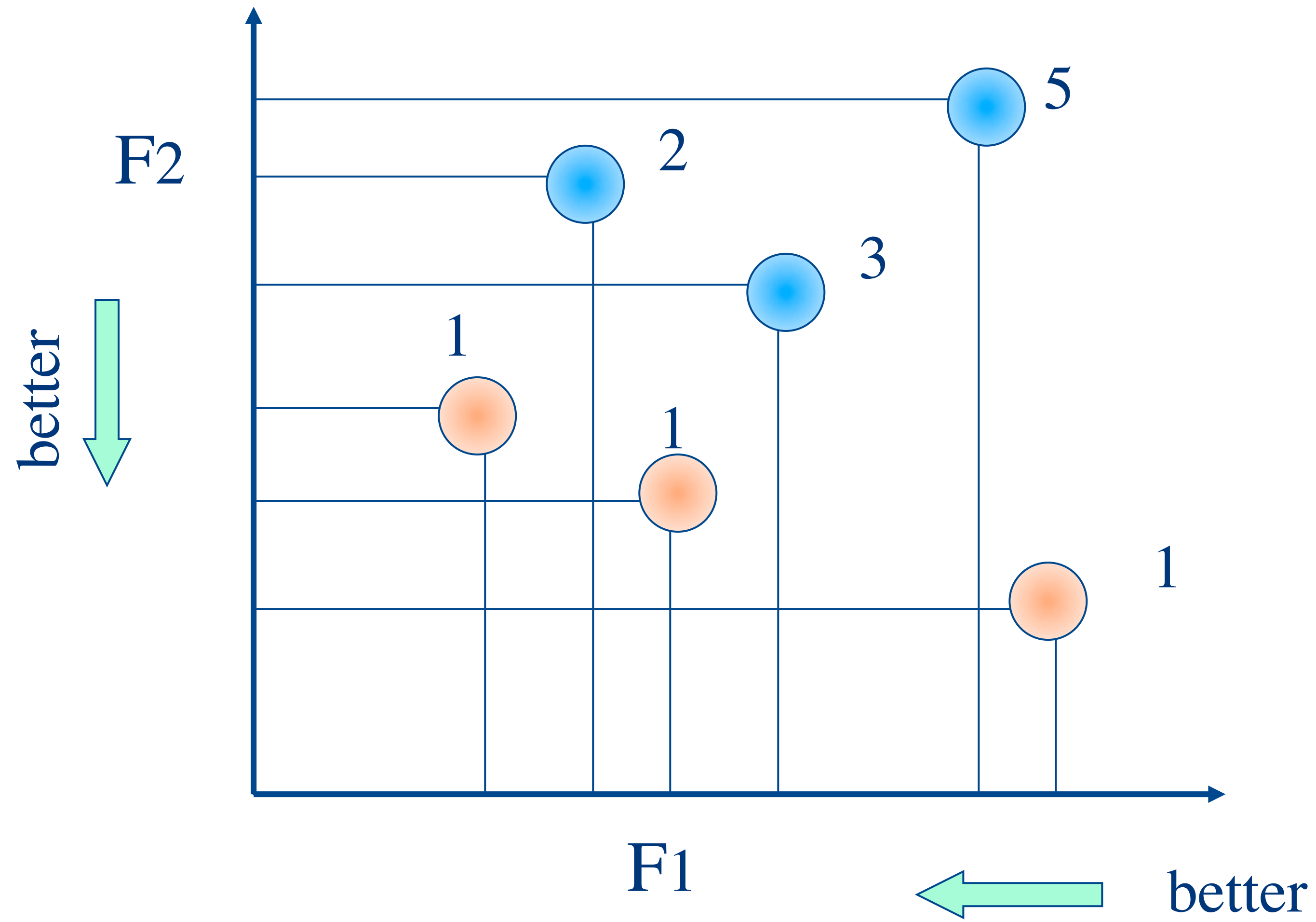
$k \ge 2$ **is number of objectives in the MOOP;**

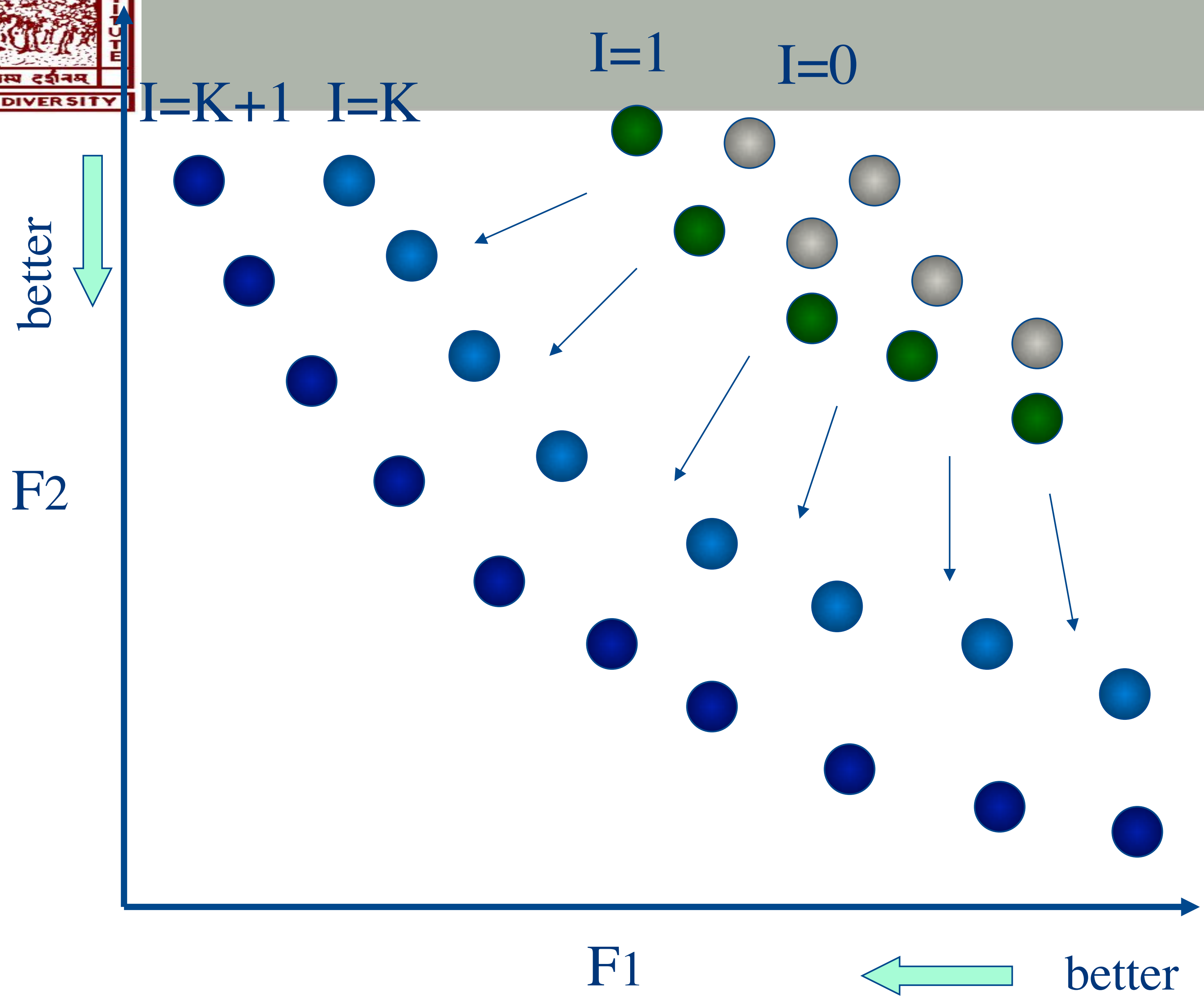
m **is the total number of constraints while** m_e **is the number of equality constraints,**

When all f_i**'s and** g_j**'s are linear, the problem is called a multi-objective linear problem (MOLP).**

If at least one of the f_i**'s is nonlinear the problem is nonlinear multi-objective optimization problem (NLMOOP).**

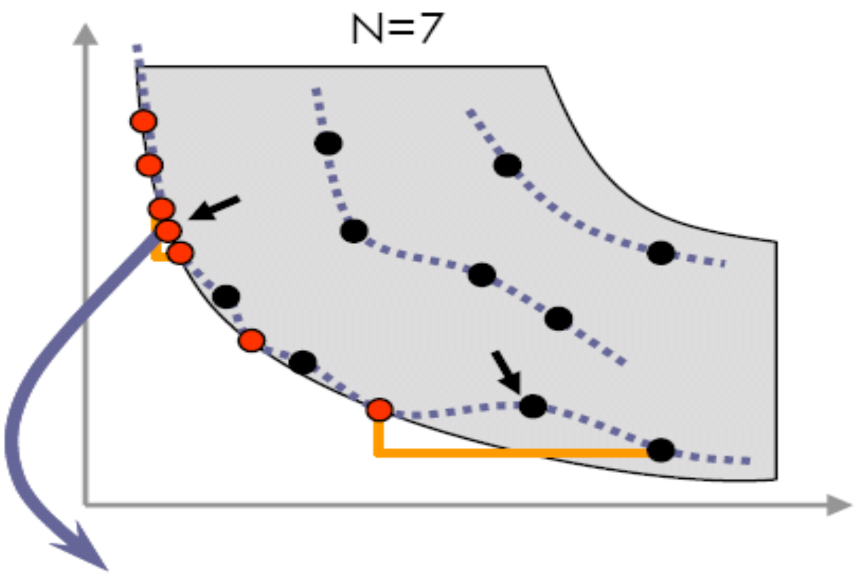
RANKING




I=1
I=0
I=K+1
I=K
better
F2
F1
better

ADVANTAGES AND DISADVANTAGES

- THE DIVESITY AMONG NON-DOMINATED SOLUTIONS IS MAINTAINED USING THE CROWDING PROCEDURE :NO EXTRA CONTROL IS NEEDED
- ELITISM PROTECS AND ALREADY FOUND PARETO-OPTIMAL SOLUTION FROM BEING DELETED
- WHEN THERE ARE MORE THAN N MEMBERS IN THE FIRST NON-DOMINATED SET, SOME PARETO-OPTIMAL SOLUTIONS MAY GIVE THEIR PLACES TO OTHER NON-PARETO-OPTIMAL SOLUTION



A Pareto-optimal solution is discarded

Thanks a lot